Regression analysis of interval data

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Standard regression models

- Suppose that we have two variables Y and X with Y being a dependent variable and X being predictor variable, related to Y according to the relation $Y = f(X)$.
- The simplest case: the linear model $Y = bX + c + \epsilon$. Here b and c are parameters and ϵ is the random errors or the noise having zero mean and the unknown variance σ^2.
- A linear regression model fits a linear function to a set of data points. When the variable X takes n specific values $x_1, ..., x_n$, the variables Y and ϵ take specific values y_i and ϵ_i, respectively, $i = 1, ..., n$, we get

$$y_i = bx_i + c + \epsilon_i, \ \ i = 1, ..., n.$$

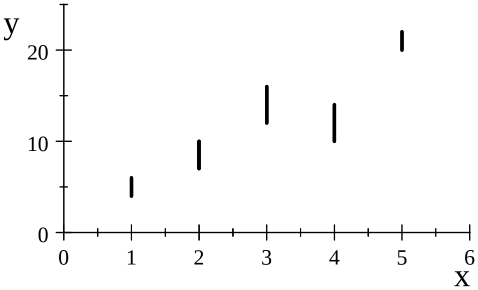
Interval data

- Suppose now that we have interval-valued observations $\mathbf{y}_i = [\underline{y}_i, \overline{y}_i]$ instead of the point-valued ones y_i, $i = 1, ..., n$.
- The simplest way is to randomly take points y_{ij} from $\mathbf{y}_i$, $i = 1, ..., n$, and to construct the j-th standard regression model with the corresponding parameters b_j and c_j. After taking M points, M regression models are constructed. Then the intervals for parameters b and c are determined as $\underline{b} = \min_j b_j$, $\overline{b} = \max_j b_j$ and $\underline{c} = \min_j c_j$, $\overline{c} = \max_j c_j$.
- The main **shortcoming** is that this way provides too wide and often non-informative intervals of the parameters.

The main idea of the proposed approach

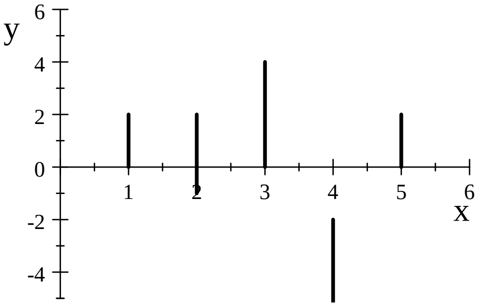
The proposed approach does not use the well-known common way for minimizing the deviation of the observed points from the "optimal" regression function f*. It maximizes the "density" or "overcrowding" of the biased intervals* $\mathbf{y}_i - f(x_i)$*. In other words, the biased intervals have to be maximally overlapped. In this case, the approach is invariant to possible changes of the interval width with changing* X*.*

Overlapping intervals (1)



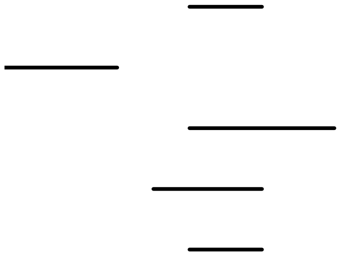
Observed intervals

Overlapping intervals (2)



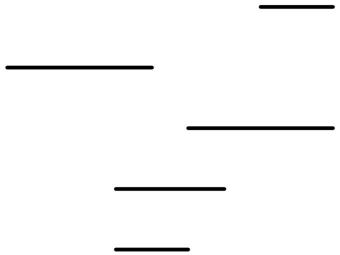
Observed intervals minus $f(x) = 4x$

Overlapping intervals (3)



Observed intervals minus $f(x) = 4x$ in another form

Overlapping intervals (4)



Observed intervals minus $f(x) = 3x$ in another form

The extended imprecise Dirichlet model

Denote

$$\underline{z}_i = \underline{y}_i - bx_i - c, \ \overline{z}_i = \overline{y}_i - bx_i - c.$$

1. Belief and plausibility functions (Dempster-Shafer theory) are

$$\mathrm{Bel}(A|\mathbf{Z}) = \frac{\sum_{i:\mathbf{z}_i \subseteq A} 1}{n}, \quad \mathrm{Pl}(A|\mathbf{Z}) = \frac{\sum_{i:\mathbf{z}_i \cap A \neq \varnothing} 1}{n}.$$

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2. Extended belief and plausibility functions with the cautious parameter s are

$$\underline{P}(A|\mathbf{Z}, s) = \frac{\sum_{i:\mathbf{z}_i \subseteq A} 1}{n+s},\quad \overline{P}(A|\mathbf{Z}, s) = \frac{s + \sum_{i:\mathbf{z}_i \cap A \neq \varnothing} 1}{n+s}.$$

The likelihood function and its maximum

The likelihood function is

$$L(\mathbf{Z}) = \Pr\left\{\underline{z}_1 \leq \epsilon_1 \leq \overline{z}_1, ..., \underline{z}_n \leq \epsilon_n \leq \overline{z}_n\right\}.$$

Proposition

If random variables $\epsilon_1, ..., \epsilon_n$ are independent, then there holds

$$\max_{\mathbf{M}} L(\mathbf{Z}) = \prod_{j=1}^{n} \left\{\overline{P}(\mathbf{z}_j|\mathbf{Z}, s) - \underline{P}(\mathbf{z}_j|\mathbf{Z}, s)\right\}.$$

The imprecise regression model

$$\max_{\mathbf{M}} L(\mathbf{Z}) = \frac{1}{(n+s)^n} \prod_{j=1}^{n} \left\{ s + \sum_{i:\mathbf{z}_i \cap \mathbf{z}_j \neq \varnothing} 1 - \sum_{i:\mathbf{z}_i \subseteq \mathbf{z}_j} 1 \right\}.$$

Parameters b and c are computed by maximizing the above function, i.e. by solving the problem

$$\prod_{j=1}^{n} \left\{ s + \sum_{i:\mathbf{z}_i \cap \mathbf{z}_j \neq \varnothing} 1 - \sum_{i:\mathbf{z}_i \subseteq \mathbf{z}_j} 1 \right\} \to \max_{b,c}.$$

Interesting properties

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3. For linear model, the parameter b does not depend on the caution parameter s.

Possible ways for computing the parameter c

1. The point-valued parameter c is computed as the mean value of $y_i' - b'x_i$, $i = 1, ..., n$. Here y_i' and b' are the middle points of intervals $\mathbf{y}_i$ and $[\underline{b}, \overline{b}]$, respectively.

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2. The interval-valued parameter:

$$\underline{c} = n^{-1} \sum_{i=1}^{n} \left(\underline{y}_i - \overline{b} x_i \right), \ \overline{c} = n^{-1} \sum_{i=1}^{n} \left(\overline{y}_i - \underline{b} x_i \right).$$

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3. Extended cautious mean values of c:

$$\underline{\mathbb{E}}_s X = (n+s)^{-1} \left(s \cdot \Omega_* + \sum_{i=1}^{n} \left(\underline{y}_i - \overline{b} x_i \right) \right),$$

$$\overline{\mathbb{E}}_s X = (n+s)^{-1} \left(s \cdot \Omega^* + \sum_{i=1}^{n} \left(\overline{y}_i - \underline{b} x_i \right) \right).$$

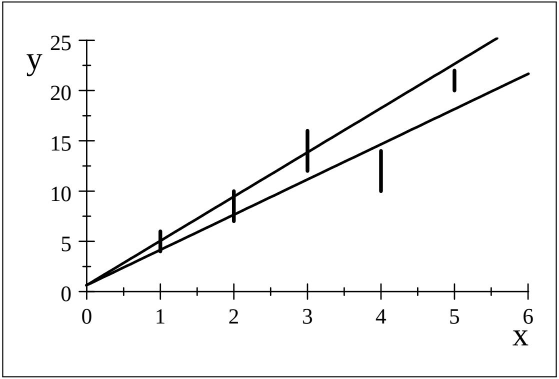
A numerical example (1)

5 pairs $(x_i, [\underline{y}_i, \overline{y}_i])$, $i = 1, ..., 5$.

i	x_i	$\underline{y}_i$	$\overline{y}_i$
1	1	4	6
2	2	7	10
3	3	12	16
4	4	10	18
5	5	20	22

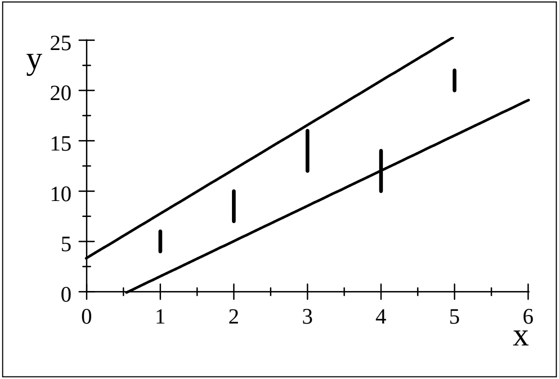
Solution 1: $\underline{b} = 3.5$, $\overline{b} = 4.4$ and $c = 0.65$.
Solution 2: $\underline{b} = 3.5$, $\overline{b} = 4.4$ and $\underline{c} = -1.97$, $\overline{c} = 3.36$.

A numerical example (2)



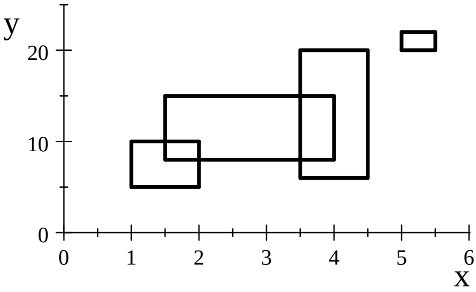
The relationship of intervals and linear functions by the first method for computing c

A numerical example (3)



The relationship of intervals and linear functions by the second method for computing c

A more general case



Observed values (x, y), $x \in [\underline{x}, \overline{x}]$, $y \in [\underline{y}, \overline{y}]$.

Questions

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