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FACULTY OF ENGINEERING

The MePiCTIr message passing algorithm:

Expert system inference in credal trees under epistemic irrelevance

Gert de Cooman

Ghent University, SYSTeMS
gert.decooman@UGent.be

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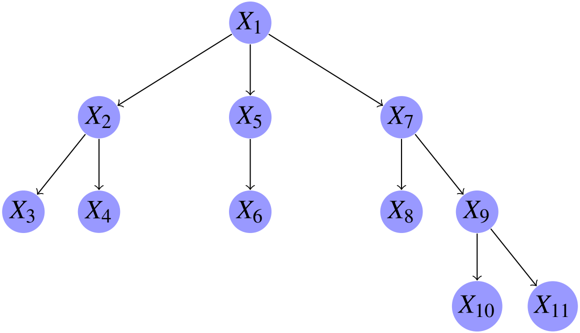
Credal networks

The special case of a tree

Basic concept

Consider a directed tree, with a variable X_s attached to each node s.



Credal trees

Local uncertainty models

Local uncertainty model associated with each node s

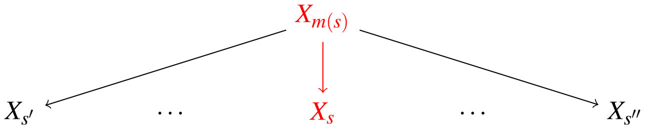
- the variable X_s may assume a value in the finite set $\mathscr{X}_s$;
- for each possible value $x_{m(s)} \in \mathscr{X}_{m(s)}$ of the mother variable $X_{m(s)}$, we have a conditional lower expectation

$$\underline{Q}_s(\cdot|x_{m(s)})\colon \mathscr{L}(\mathscr{X}_s) \to \mathbb{R}$$

where

$\underline{Q}_s(f|x_{m(s)}) =$ lower expectation of $f(X_s)$, given that $X_{m(s)} = x_{m(s)}$

- local model $\underline{Q}_s(\cdot|X_{m(s)})$ is a conditional lower expectation operator

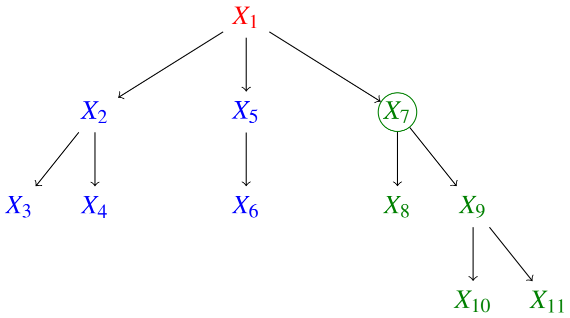


Credal trees under epistemic irrelevance

Definition

The graphical structure is interpreted as follows:

Conditional on the mother variable, the non-parent non-descendants of each node variable are epistemically irrelevant to it and its descendants.



Credal networks under epistemic irrelevance

As an expert system

When the credal network is a (Markov) tree we can find the joint model from the local models recursively, from leaves to root.

Exact message passing algorithm

- credal tree treated as an expert system
- linear complexity in the number of nodes

Python code

- written by Filip Hermans
- testing and connection with strong independence in cooperation with Marco Zaffalon and Alessandro Antonucci

Current (toy) applications in HMMs

- character recognition
- air traffic trajectory tracking and identification

Constructing global from local models

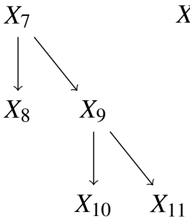
From local to global models

For each node s, we want to construct a global model

$$\underline{P}_s(\cdot|X_{m(s)})\colon \mathscr{L}(\mathscr{X}_{\downarrow s}) \to \mathbb{R}$$

representing the uncertainty about all variables $X_{\downarrow s}$ in the subtree below s:

$$\downarrow s = \{t\colon t \text{ lies below } s\}$$



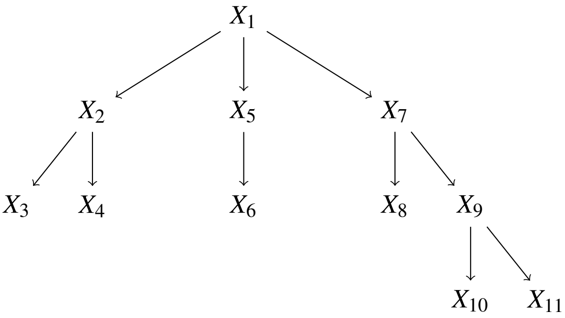
$$X_{\downarrow 7} = (X_7, X_8, X_9, X_{10}, X_{11})$$

Constructing global from local models

How to construct the global models

First crucial observation

We can build up any tree in a recursive fashion, repeating much simpler basic building blocks.



Constructing global from local models

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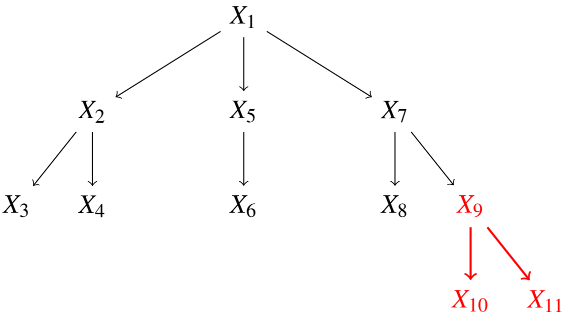

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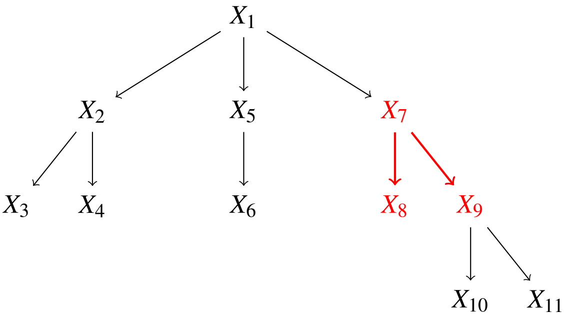

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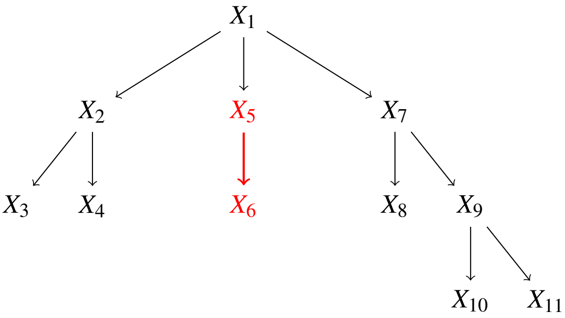

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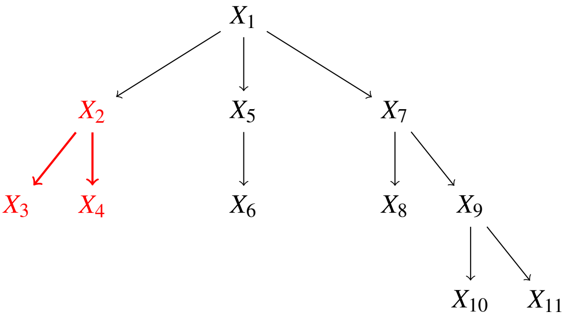

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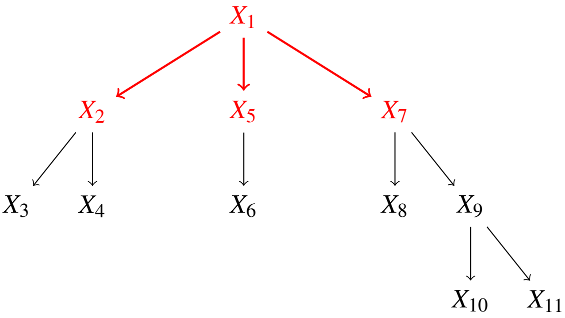
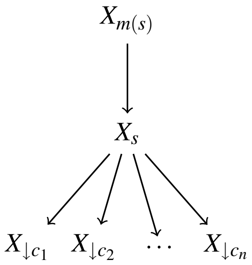


Constructing global from local models

Recursive construction of the joint: graphical representation



Constructing global from local models

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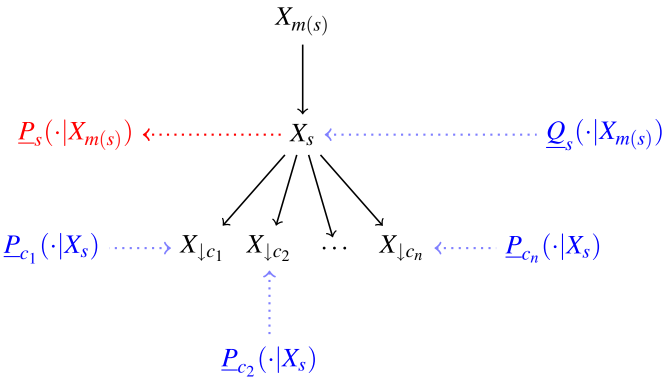
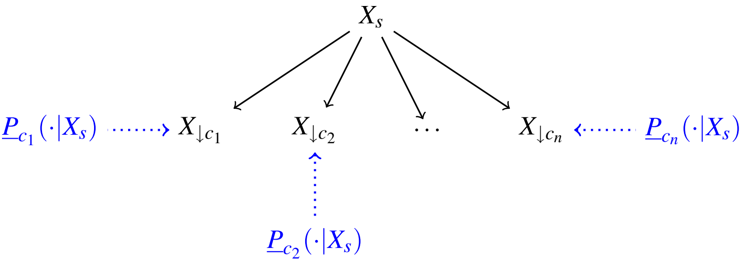


Constructing global from local models

Recursive construction of the joint: epistemically independent product

Conditional on X_s, the child variables X_c, $c \in C(s)$ are epistemically independent.

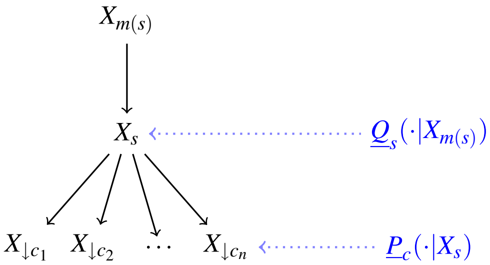


Epistemically independent product on $\mathscr{L}(X_{\downarrow s \setminus \{s\}})$:

$$\underline{E}_s(\cdot|X_s) = \otimes_{c \in C(s)} \underline{P}_c(\cdot|X_s)$$

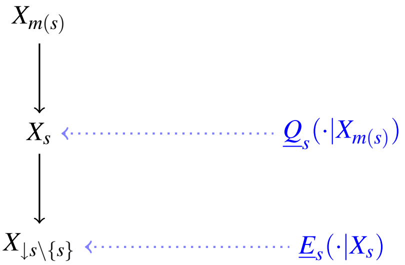
Constructing global from local models

Recursive construction of the joint: marginal extension



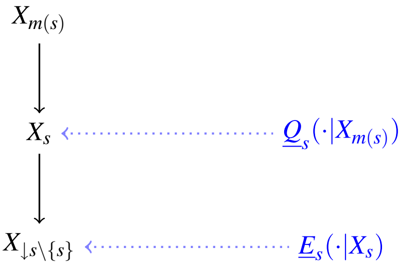
Constructing global from local models

Recursive construction of the joint: marginal extension



Constructing global from local models

Recursive construction of the joint: marginal extension



Marginal extension or Law of iterated expectation:

$$\underline{P}_s(\cdot|X_{m(s)}) = \underline{Q}_s(\underline{E}_s(\cdot|X_s)|X_{m(s)}) = \underline{Q}_s(\otimes_{c\in C(s)}\underline{P}_c(\cdot|X_s)|X_{m(s)})$$

Constructing global from local models

Recursive construction of the joint: marginal extension

Marginal extension or Law of iterated expectation:

$$\underline{P}_s(\cdot|X_{m(s)}) = \underline{Q}_s(\underline{E}_s(\cdot|X_s)|X_{m(s)}) = \underline{Q}_s(\otimes_{c\in C(s)}\underline{P}_c(\cdot|X_s)|X_{m(s)})$$

Constructing global from local models

Fundamental result

We start at the leaves with:

$$\underline{P}_t(\cdot|X_{m(t)}) = \underline{Q}_t(\cdot|X_{m(t)}) \text{ for all leaves } t$$

and move recursively upwards to the root of the tree.

Theorem

In this recursive fashion, we construct the point-wise smallest (most conservative) collection of global models $\underline{P}_s(\cdot|X_{m(s)})$ that

(i) are coherent;

(ii) are compatible with the local information;

(iii) reflect all the epistemic irrelevancies embodied in the graphical structure.

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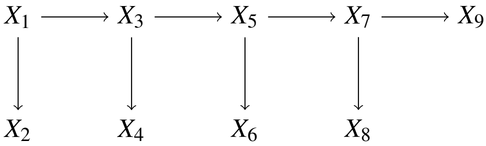
Explaining the basics of the algorithm

An example



Explaining the basics of the algorithm

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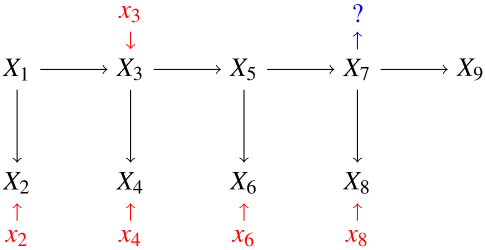

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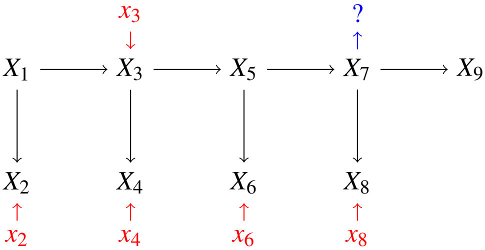
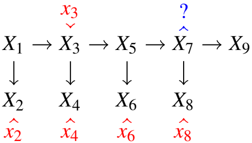


Regular extension

$$
\begin{aligned}
&\underline{R}(g(X_7)|X_2 = x_2, X_3 = x_3, X_4 = x_4, X_6 = x_6, X_8 = x_8) \\
&\qquad = \max\left\{\mu \in \mathbb{R} \colon \underline{P}_1\left(I_{\{x_2\}} I_{\{x_3\}} I_{\{x_4\}} I_{\{x_6\}} I_{\{x_8\}} [g(X_7) - \mu]\right)\right\}
\end{aligned}
$$

Explaining the basics of the algorithm

An example



We have to calculate $\underline{P}_1(f_1)$, where

$$f_1 = I_{\{x_2\}} I_{\{x_3\}} I_{\{x_4\}} I_{\{x_6\}} I_{\{x_8\}} [g(X_7) - \mu] = I_{\{x_2\}} f_3$$

Now use the recursion formula for $\underline{P}_1$:

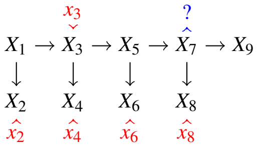
$$\underline{P}_1(f_1) = \underline{Q}_1(\underline{E}_1(f_1|X_1))$$

and the factorisation property of independent natural extension:

$$\underline{E}_1(f_1|X_1) = \underline{E}_1(I_{\{x_2\}} f_3|X_1) = \overline{\underline{Q}}_2(\{x_2\}|X_1) \odot \underline{P}_3(f_3|X_1)$$

Explaining the basics of the algorithm

An example



We have to calculate $\underline{P}_3(f_3|X_1)$, where

$$f_3 = I_{\{x_3\}}I_{\{x_4\}}I_{\{x_6\}}I_{\{x_8\}}[g(X_7)-\mu] = I_{\{x_3\}}I_{\{x_4\}}f_5$$

Now use the recursion formula for $\underline{P}_3(\cdot|X_1)$:

$$\underline{P}_3(f_3|X_1) = \underline{Q}_3(\underline{E}_3(f_3|X_3)|X_1)$$

and the factorisation property of independent natural extension:

$$\underline{E}_3(f_3|X_3) = \underline{E}_3(I_{\{x_3\}}I_{\{x_4\}}f_5|X_3) = I_{\{x_3\}}\overline{Q}_4(\{x_4\}|x_3)\odot\underline{P}_5(f_5|x_3)$$

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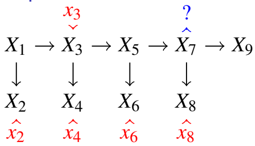
Explaining the basics of the algorithm

An example



In summary, so far we have:

$$\underline{P}_3(f_3|X_1) = \overline{\underline{Q}}_3(\{x_3\}|X_1) \odot \overline{\underline{Q}}_4(\{x_4\}|x_3) \odot \underline{P}_5(f_5|x_3)$$

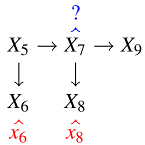
$$\underline{E}_1(f_1|X_1) = \overline{\underline{Q}}_2(\{x_2\}|X_1) \odot \overline{\underline{Q}}_3(\{x_3\}|X_1) \odot \overline{\underline{Q}}_4(\{x_4\}|x_3) \odot \underline{P}_5(f_5|x_3)$$

$$\underline{P}_1(f_1) = \begin{cases} \underline{Q}_1\left(\underline{Q}_2(\{x_2\}|X_1)\underline{Q}_3(\{x_3\}|X_1)\right)\underline{Q}_4(\{x_4\}|x_3)\underline{P}_5(f_5|x_3) & \underline{P}_5(f_5|x_3) \geq 0 \\ \overline{Q}_1\left(\overline{Q}_2(\{x_2\}|X_1)\overline{Q}_3(\{x_3\}|X_1)\right)\overline{Q}_4(\{x_4\}|x_3)\underline{P}_5(f_5|x_3) & \underline{P}_5(f_5|x_3) \leq 0 \end{cases}$$

$$= \overline{\underline{P}}_1(\{(x_2,x_3,x_4)\}) \odot \underline{P}_5(f_5|x_3)$$

Explaining the basics of the algorithm

An example



Because $\underline{P}_1(f_1) \geq 0 \Leftrightarrow \underline{P}_5(f_5|x_3) \geq 0$, we have to calculate $\underline{P}_5(f_5|x_3)$, where

$$f_5 = I_{\{x_6\}} I_{\{x_8\}}[g(X_7) - \mu] = I_{\{x_6\}} f_7$$

Now use the recursion formula for $\underline{P}_5(\cdot|x_3)$:

$$\underline{P}_5(f_5|x_3) = \underline{Q}_5(\underline{E}_5(f_5|X_5)|x_3)$$

and the factorisation property of independent natural extension:

$$\underline{E}_5(f_5|X_5) = \underline{E}_5(I_{\{x_6\}} f_7|X_5) = \underline{Q}_6(\{x_6\}|X_5) \odot \underline{P}_7(f_7|X_5)$$

Explaining the basics of the algorithm

An example

$$\begin{array}{ccccc} & & ? & & \\ & & \hat{} & & \\ X_5 & \to & X_7 & \to & X_9 \\ \downarrow & & \downarrow & & \\ X_6 & & X_8 & & \\ \hat{x}_6 & & \hat{x}_8 & & \end{array}$$

We have to calculate $\underline{P}_7(f_7|X_5)$, where

$$f_7 = I_{\{x_8\}}[g(X_7) - \mu]$$

Now use the recursion formula for $\underline{P}_7(\cdot|X_5)$:

$$\underline{P}_7(f_7|X_5) = \underline{Q}_7(\underline{E}_7(f_7|X_7)|X_5)$$

and apply independent natural extension:

$$\underline{E}_7(I_{\{x_8\}}[g(X_7) - \mu]|X_7) = \overline{\underline{Q}}_8(I_{\{x_8\}}|X_7) \odot [g(X_7) - \mu]$$

Explaining the basics of the algorithm

An example

$$
\begin{array}{ccccc}
 & & ? & & \\
 & & \wedge & & \\
X_5 & \rightarrow & X_7 & \rightarrow & X_9 \\
\downarrow & & \downarrow & & \\
X_6 & & X_8 & & \\
\wedge & & \wedge & & \\
x_6 & & x_8 & &
\end{array}
$$

In summary, we have that

$$\underline{P}_1(f_1) \geq 0 \Leftrightarrow \underline{P}_5(f_5|x_3) \geq 0$$

where

$$\underline{P}_5(f_5|x_3) = \underline{Q}_5(\underline{Q}_6(\{x_6\}|X_5) \odot \underline{P}_7(f_7|X_5)|x_3)$$

$$\underline{P}_7(f_7|X_5) = \underline{Q}_7(\overline{\underline{Q}}_8(I_{\{x_8\}}|X_7) \odot [g(X_7) - \mu]|X_5)$$

Explaining the basics of the algorithm

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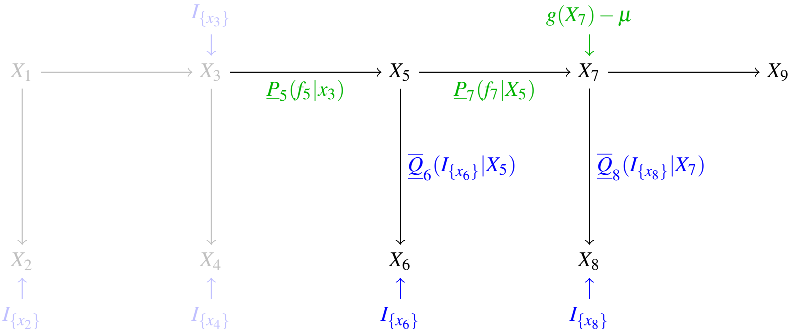

$$\underline{P}_7(f_7|X_5) = \underline{Q}_7(\overline{Q}_8(I_{\{x_8\}}|X_7) \odot [g(X_7)-\mu]|X_5)$$

$$\underline{P}_5(f_5|x_3) = \underline{Q}_5(\underline{Q}_6(\{x_6\}|X_5) \odot \underline{P}_7(f_7|X_5)|x_3)$$

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Conclusion

On the positive side

- very efficient, essentially linear in number of nodes
- not much more conservative than strong independence:
  - the same for forward inference
  - dilation for backward inference

On the negative side

- the algorithm works only for trees, not for polytrees or more general acyclic directed nets
- the algorithm works only for one function g at a time, it doesn't provide the entire credal set

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