Set-valued stochastic processes in vibration analysis

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joint work with Christoph Adam and Michael Oberguggenberger

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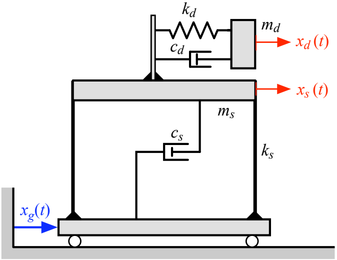
Issue

Consider slender structure

- modelled by single-degree-of-freedom (SDOF) oscillator with mass m_s, viscous damping parameter c_s and stiffness k_s



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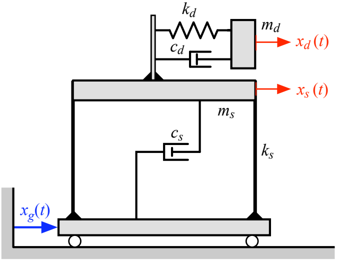

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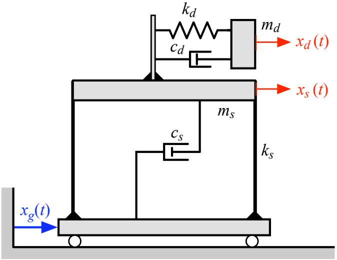
- ground excitation x_g (earthquake)



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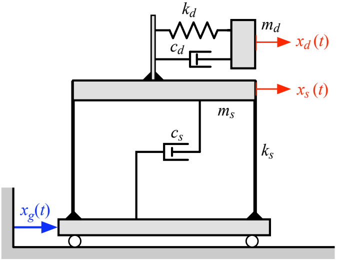
- modelled by single-degree-of-freedom (SDOF) oscillator with mass m_s, viscous damping parameter c_s and stiffness k_s
- weak damping
- ground excitation x_g (earthquake)
- leads to structural vibrations with large amplitudes (displacement x_s)



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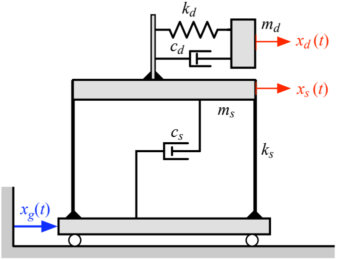

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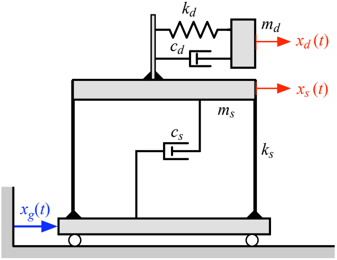

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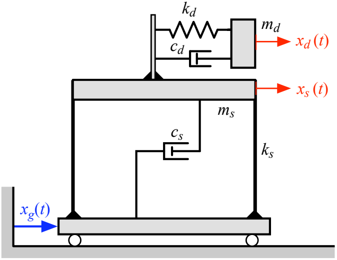

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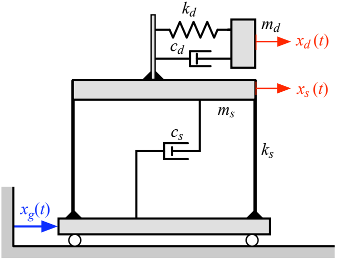
- kinetic energy is transferred from structure to TMD
- optimal tuning of TMD depending on type of excitation
- possible in theory, very difficult in practice $\rightarrow$ parameter uncertainty



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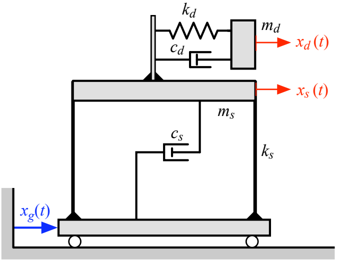
Combined structure-TMD system

- modelled by two-degrees-of-freedom oscillator
  $\longrightarrow$ system of ODEs



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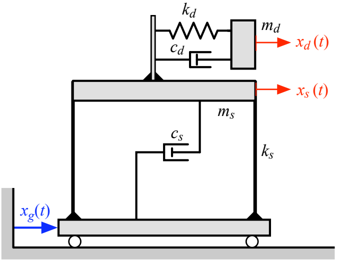
- modelling ground acceleration $\ddot{x}_g$ by white noise
  $\longrightarrow$ system of stochastic differential equations



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- modelled by two-degrees-of-freedom oscillator
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- modelling ground acceleration $\ddot{x}_g$ by white noise
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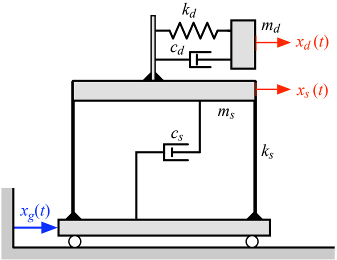
- modelling parameter uncertainty by random sets
  $\longrightarrow$ set-valued processes as output



Stochastic differential equations

$$dx_t = f(t, x_t)dt + G(t, x_t)dw_t$$

with

- $t_0 \leq t \leq \overline{t} < \infty$,
- $\{w_t\}_{t \geq t_0}$ being an m-dimensional Wiener process on a probability space $(\Omega_w, \Sigma_w, P_w)$,
- initial value x_{t_0} and coefficients f and G:

$$\begin{aligned} x_{t_0} &: \quad \Omega_w \to \mathbb{R}^d, & \omega_w &\mapsto x_{t_0}(\omega_w), \\ f &: \quad [t_0, \overline{t}] \times \mathbb{R}^d \to \mathbb{R}^d, & (t, x) &\mapsto f(t, x), \\ G &: \quad [t_0, \overline{t}] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}, & (t, x) &\mapsto G(t, x). \end{aligned}$$

Stochastic differential equations

$$dx_{t,a} = f(t, a, x_{t,a})dt + G(t, a, x_{t,a})dw_t$$

with uncertain parameters $a \in \mathbb{A} \subseteq \mathbb{R}^p$ and

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$$\begin{array}{rcll} x_{t_0} & : & \mathbb{A} \times \Omega_w \to \mathbb{R}^d, & (a, \omega_w) \mapsto x_{t_0,a}(\omega_w), \\ f & : & [t_0, \overline{t}] \times \mathbb{A} \times \mathbb{R}^d \to \mathbb{R}^d, & (t, a, x) \mapsto f(t, a, x), \\ G & : & [t_0, \overline{t}] \times \mathbb{A} \times \mathbb{R}^d \to \mathbb{R}^{d \times m}, & (t, a, x) \mapsto G(t, a, x). \end{array}$$

Solution processes

- Assume that for each $a \in \mathbb{A}$ the conditions for the existence of a unique solution $\{x_{t,a}\}_{t \in [t_0, \overline{t}]}$ are fulfilled.

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which is a stochastic process on $[t_0, \overline{t}] \times \mathbb{A}$ and $(\Omega_w, \Sigma_w, P_w)$.

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  which is a stochastic process on $[t_0, \overline{t}] \times \mathbb{A}$ and $(\Omega_w, \Sigma_w, P_w)$.
- Under certain conditions it can be shown that x has a version
  - whose sample paths are continuous on $[t_0, \overline{t}] \times \mathbb{A}$,
  - which is $\mathcal{B}([t_0, \overline{t}]) \otimes \mathcal{B}(\mathbb{A}) \otimes \Sigma_w$-measurable.

Parameter uncertainty

- To model the uncertainty of a we use a random compact set

$$A : \Omega_{\mathbb{A}} \to \mathcal{K}'(\mathbb{A})$$

– on a probability space $(\Omega_{\mathbb{A}}, \Sigma_{\mathbb{A}}, P_{\mathbb{A}})$

– with non-empty compact values being subsets of $\mathbb{A}$

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- Defining measurability condition: for all $B \in \mathcal{B}(\mathbb{R}^d)$ it holds that

$$A^{-}(B) = \{\omega_{\mathbb{A}} : A(\omega_{\mathbb{A}}) \cap B \neq \emptyset\} \in \Sigma_{\mathbb{A}}.$$

Set-valued solution process

- Define the set-valued map

$$X : (t, \omega) \mapsto \{x_{t,a}(\omega_w) : a \in A(\omega_{\mathbb{A}})\}$$

where $(t, \omega) \in [t_0, \overline{t}] \times \Omega$ and Ω denotes the product space

$$(\Omega, \Sigma, P) = (\Omega_{\mathbb{A}} \times \Omega_w, \Sigma_{\mathbb{A}} \otimes \Sigma_w, P_{\mathbb{A}} \otimes P_w).$$

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- **Is this a set-valued process (measurability)?**
- **What are its properties?**

Set-valued solution process

- X is a set-valued process on $[t_0, \overline{t}]$ and the completed probability space $(\Omega, \overline{\Sigma}^P, \overline{P})$ with values in $\mathcal{K}'(\mathbb{R}^d)$, i.e., for all $t \in [t_0, \overline{t}]$ and $B \in \mathcal{B}(\mathbb{R}^d)$ it holds that

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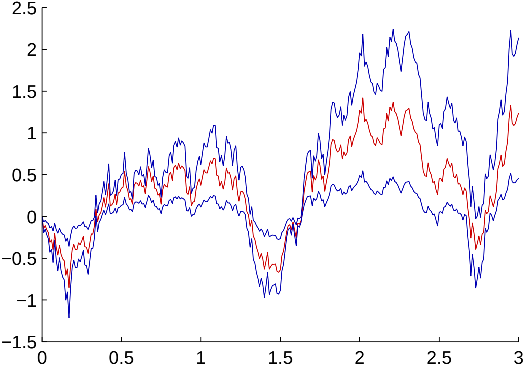
- all sample functions of X are continuous with respect to the Hausdorff-metric on $\mathcal{K}'(\mathbb{R}^d)$,
- X is measurable with respect to the product-σ-algebra $\mathcal{B}([t_0, \overline{t}]) \otimes \overline{\Sigma}^P$.

Picture of a sample path



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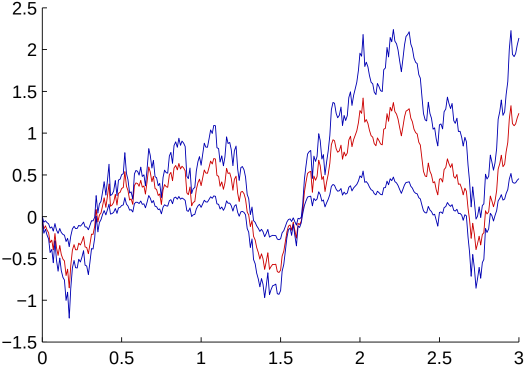

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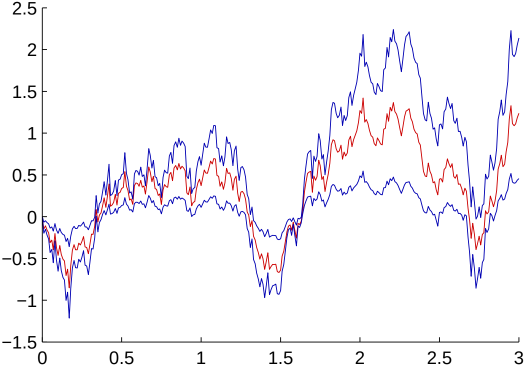
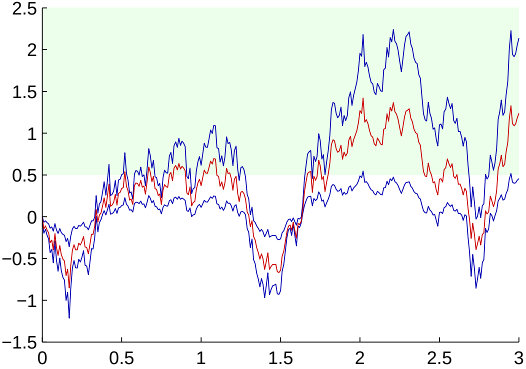


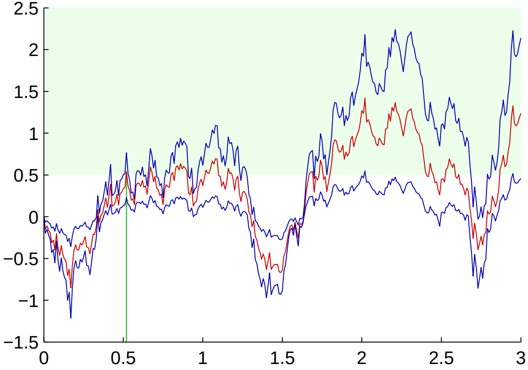
- blue lines: boundaries of sample path of set-valued process X
- red line: sample path of a selection ξ

First entrance and inclusion times



When does ξ enter B for the first time?

First entrance and inclusion times



When does ξ enter B for the first time?
$\rightarrow$ first entrance time $\tau_\xi^B : \omega \mapsto \inf\{t : \xi_t(\omega) \in B\}$

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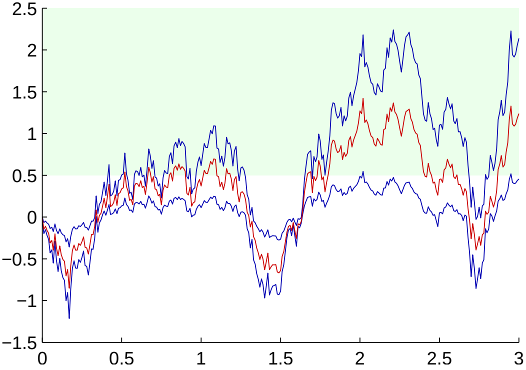

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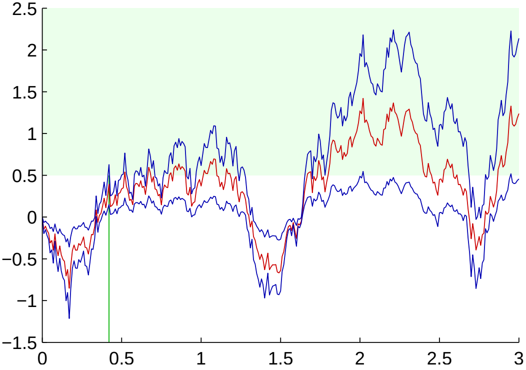

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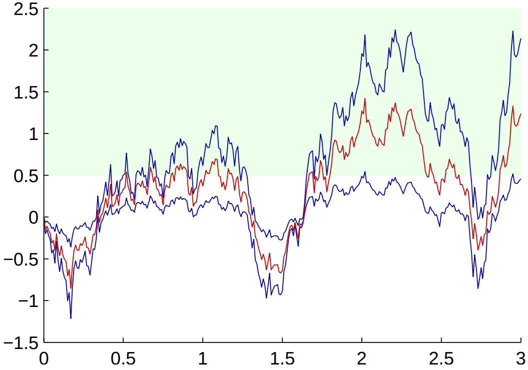

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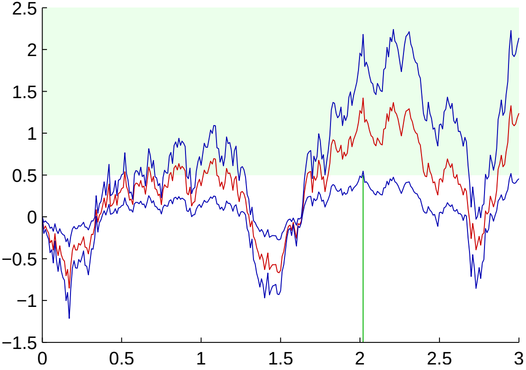

When is X contained in B for the first time?
$\rightarrow$ first inclusion time $\overline{\tau}^{B} : \omega \mapsto \inf\{t : X_t(\omega) \subseteq B\}$

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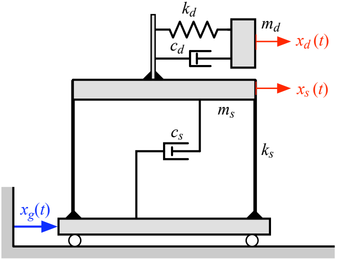
- can be attained by first entrance times of selection processes of X.

TMD - Mechanical model



Defining

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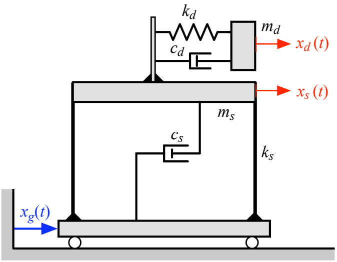

Defining

- mass ratio: $\mu = \frac{m_d}{m_s} \ll 1$

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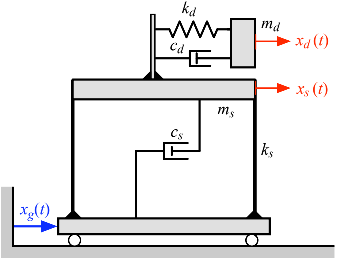

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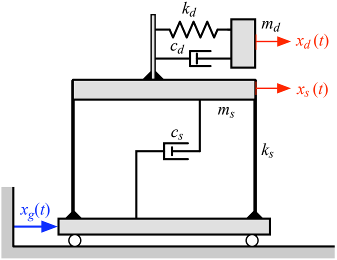

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- non-dimensional damping coefficients: $\zeta_s = \frac{c_s}{2\omega_s m_s}$, $\zeta_d = \frac{c_d}{2\omega_d m_d}$

TMD - Mechanical model

The coupled equations of motion can be written as a 4-dimensional linear system of SDEs of first order:

$$dy_t = M\, y_t\, dt + G\, dw_t, \quad y_0 = 0$$

where

$$y_t = \begin{pmatrix} x_s \\ x_d \\ \dot{x}_s \\ \dot{x}_d \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_s^2 - \omega_d^2 \mu & \omega_d^2 \mu & -2\zeta_s \omega_s - 2\zeta_d \omega_d \mu & 2\zeta_d \omega_d \mu \\ \omega_d^2 & -\omega_d^2 & 2\zeta_d \omega_d & -2\zeta_d \omega_d \end{pmatrix}$$

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- optimal values $\overline{a} = (\overline{\zeta}_d, \overline{\omega}_d)$ under white noise excitation:

$$\overline{\zeta}_d = \sqrt{\frac{\mu(1-\mu/4)}{4(1+\mu)(1-\mu/2)}}, \quad \overline{\omega}_d = \frac{\sqrt{1-\mu/2}}{1+\mu} \cdot \omega_s$$

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- Assessment for both parameters: The probability that the parameter differs from its optimal value by more than 40% is at most 0.01.

$$P_i(|a_i - \overline{a}_i| > 0.4\overline{a}_i) \leq 0.01$$

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Modelling uncertainty by Tchebychev random sets

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- Tchebychev's inequality: $P(|a_i - \mathrm{E}(a_i)| > c) \leq \frac{\mathrm{V}(a_i)}{c^2}$

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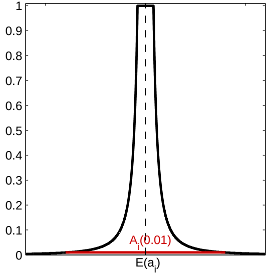
Modelling uncertainty by Tchebychev random sets

- Tchebychev's inequality: $P(|a_i - \mathrm{E}(a_i)| > c) \leq \frac{\mathrm{V}(a_i)}{c^2}$
- implies random set

$$A_i : (0,1] \to \mathcal{K}'(\mathbb{R}), c \mapsto \left[\mathrm{E}(a_i) - \sqrt{\frac{\mathrm{V}(a_i)}{c}}, \mathrm{E}(a_i) - \sqrt{\frac{\mathrm{V}(a_i)}{c}}\right]$$



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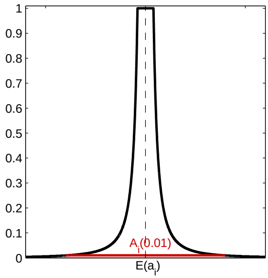
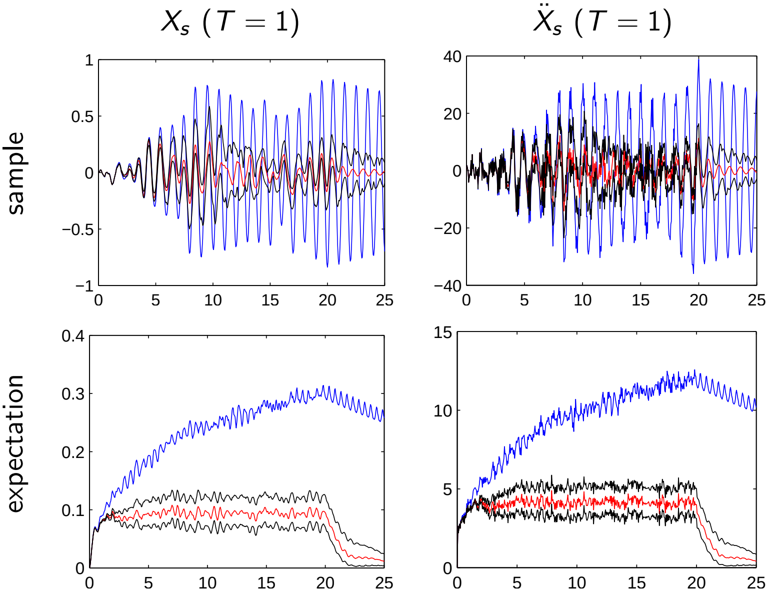
- Random sets for $a_1 = \zeta_d$ and $a_2 = \omega_d$ are combined by random set independence to one random set A on $\Omega_{\mathbb{A}} = (0,1]^2$ with values in $\mathcal{K}'(\mathbb{R}^2)$



TMD - Output: displacement and acceleration



TMD - Output: first entrance and inclusion times

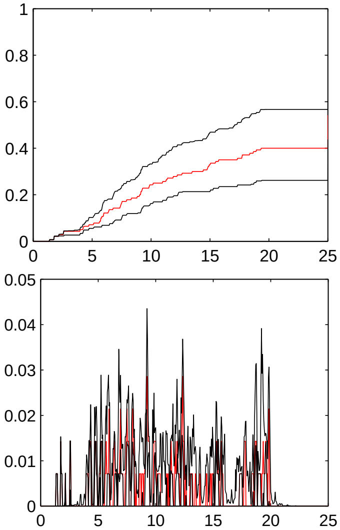
$b = 0.5 \cdot \mathrm{E}(\sup_{t \in [t_0, \overline{t}]} |\tilde{x}_t|)$
$B = [b, \infty)$

CDFs of $\underline{\tau}^B$, $\overline{\tau}^B$ and $\tau^B_{x_{\cdot}, \overline{a}}$

$\overline{P}_t(B) = P(X_t \cap B \neq \emptyset)$

$\underline{P}_t(B) = P(X_t \subseteq B)$

$P_{\overline{a},t}(B) = P(x_{t,\overline{a}} \in B)$



TMD - Output: reduction of displacement

Reduction of displacement compared to displacement $\tilde{x}_s$ of structure without TMD:

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- peak displacement: $r_{\infty,a}(\omega_w) = \frac{\sup_{t\in[0,\bar{t}]} |x_{s,t,a}(\omega_w)|}{\sup_{t\in[0,\bar{t}]} |\tilde{x}_{s,t}(\omega_w)|}$

TMD - Output: reduction of displacement

Reduction of displacement compared to displacement $\tilde{x}_s$ of structure without TMD:

- peak displacement: $r_{\infty,a}(\omega_w) = \frac{\sup_{t\in[0,\bar{t}]} |x_{s,t,a}(\omega_w)|}{\sup_{t\in[0,\bar{t}]} |\tilde{x}_{s,t}(\omega_w)|}$
- mean square reduction: $r_{2,a}(\omega_w) = \sqrt{\frac{\int_0^{\bar{t}} |x_{s,t,a}(\omega_w)|^2 dt}{\int_0^{\bar{t}} |\tilde{x}_{s,t}(\omega_w)|^2 dt}}$

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- two possible set-valued analogues:

$$
\begin{aligned}
R_*^{(1)}(\omega) &= \{r_{*,a}(\omega_w) : a \in A(\omega_{\mathbb{A}})\} \\
R_*^{(2)}(\omega) &= [\underline{R}_*(\omega), \overline{R}_*(\omega)]
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- it holds that $R_*^{(1)} \subseteq R_*^{(2)}$

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