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FACULTY OF ENGINEERING

Asymptotic properties of imprecise Markov chains

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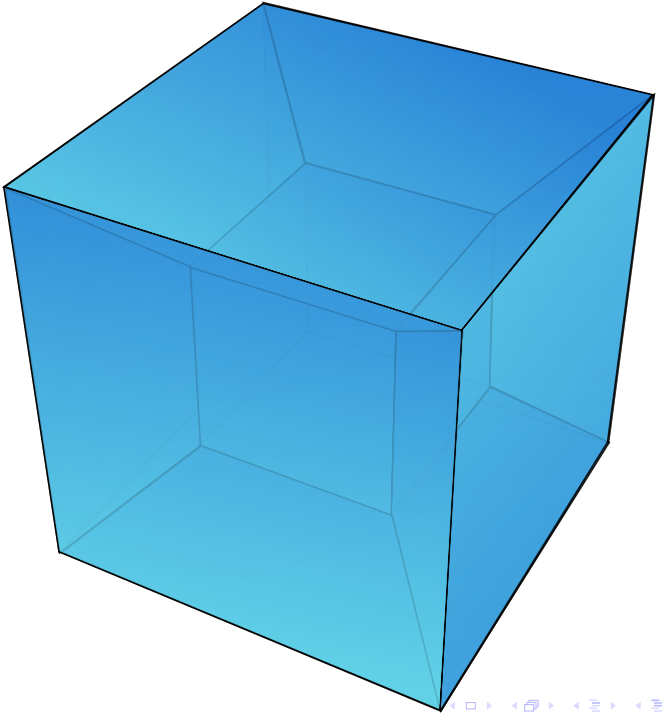

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$$\overline{T}f := \begin{pmatrix} \overline{P}\left(f|x_1\right) \\ \overline{P}\left(f|x_2\right) \\ \vdots \\ \overline{P}\left(f|x_n\right) \end{pmatrix} \quad \text{so,} \quad \overline{T}f(x_i) = \overline{P}\left(f|x_i\right).$$

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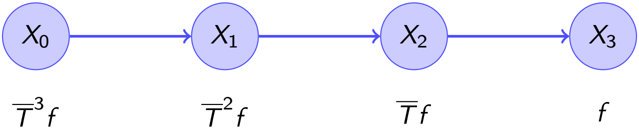

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- Doing so, $\overline{P}(f) = \overline{P}_0(\overline{T}^3 f)$ where $\overline{T}^3 := \overline{T} \circ \overline{T} \circ \overline{T}$.

Outline

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$\overline{T}$ has all the usual properties

The properties of upper previsions are transfered to the transition operator

1. If $f \leq g$ then $\overline{T}f \leq \overline{T}g$ [monotonicity preserving],
2. If $c \in \mathbb{R}$ then $\overline{T}(f + c) = \overline{T}f + c$ [constant additivity],
3. If $\alpha > 0$ then $\overline{T}(\alpha f) = \alpha\overline{T}f$ [positive-homogeneity].

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From the properties, it automatically follows that

- the map $\overline{T}$ is bounded,
- the map $\overline{T}$ is non-expansive

$$\|\overline{T}f - \overline{T}g\|_\infty \leq \|f - g\|_\infty.$$

If a map has the first two properties it is also called topical.

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A lot is known for bounded and non-expansive maps

1. Edelstein [1963] showed that all elements of the ω-limit set of f, $\omega_{\overline{T}}(f)$, are recurrent and moreover, that $\overline{T}$ acts isometrically on every ω-limit set.

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2. Due to a result of Sine [1990] we know that $\overline{T}^k f$ converges to a limit cycle for $k \to \infty$,

$$\lim_{k \to \infty} \overline{T}^k f = \xi_f \text{ where } \overline{T}^p \xi_f = \xi_f.$$

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3. Lemmens and Scheutzow [2005] showed for topical functions that the maximal period for a periodic point is $\binom{n}{\lfloor n/2 \rfloor}$.

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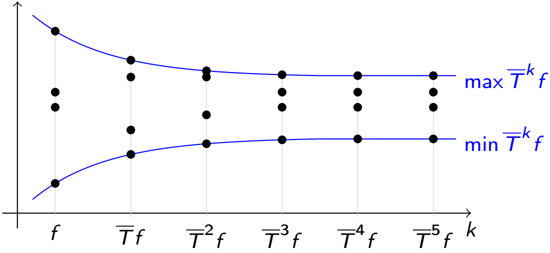
How we interprete convergence



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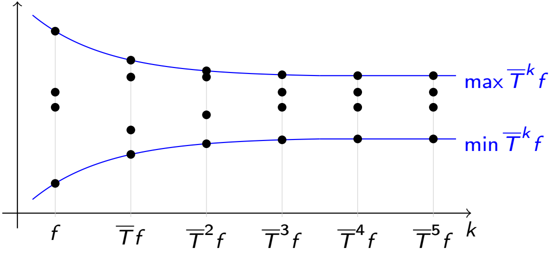

1. An imprecise Markov chain PF-converges if $\max \overline{T}^k f \to \min \overline{T}^k f$ as $k \to \infty$.
2. An imprecise Markov chain converges if $\overline{T}^k f \to \xi_f$ with $\overline{T}\xi_f = \xi_f$ as $k \to \infty$

Clearly PF-convergence $\Rightarrow$ convergence.

Convergence can be restated in terms of fixed points

It can be easily seen that

Proposition

1. *An imprecise Markov chain converges if and only if every periodic points is also a fixed point.*

Remember that f is a fixed point if $Tf = f$.

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To conclude upon (PF)-convergence, all periodic points of $\overline{T}$ must be investigated. This is easy in the linear case, however ...

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An accessibility relation $\rightarrow$ can be defined

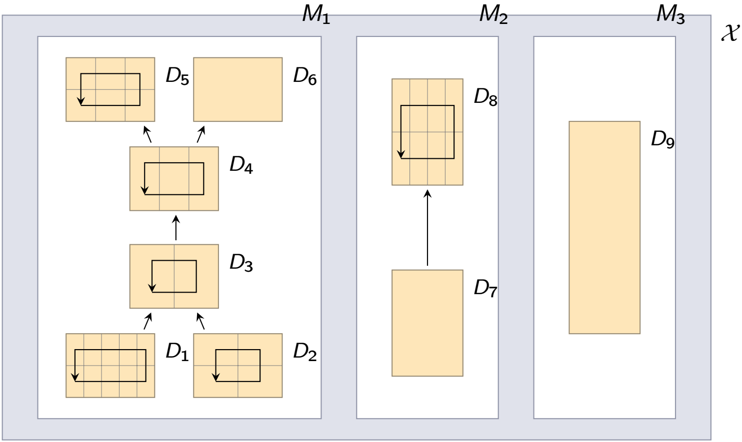
Definition

We say that state y is accessible from state x, $x \rightarrow y$, if there exists $n \in \mathbb{N}$ such that

$$\overline{T}^n I_y(x) > 0.$$

- As $\rightarrow$ is reflexive and transitive it determines a preorder on $\mathcal{X}$.
- The binary relation $\leftrightarrow$ on $\mathcal{X}$ is the associated equivalence relation.
- This communication relation $\leftrightarrow$ partitions the state set $\mathcal{X}$ into equivalence classes called communication classes.
- The preorder $\rightarrow$ induces a partial order on this partition, also denoted by $\rightarrow$.
- Because of this partial order $\rightarrow$, maximal communication classes will exist.

The relation $\to$ structures the state space



Communication classes split up the imprecise Markov chain

Proposition

Consider a stationary imprecise Markov chain with upper transition operator $\overline{\mathrm{T}}$. Let C be a closed set of states, and let $\mathcal{C}$ be a partition of the state set $\mathcal{X}$ into closed sets. Then

1. $\overline{\mathrm{T}}(hI_B)(x) = 0$ *for all* $h \in \mathcal{L}(\mathcal{X})$, *all* $x \in C$ *and all* $B \subseteq C^c$;
2. $\overline{\mathrm{T}}h(x) = \overline{\mathrm{T}}(hI_C)(x)$ *for all* $h \in \mathcal{L}(\mathcal{X})$ *and all* $x \in C$;
3. $\overline{\mathrm{T}}h = \sum_{C \in \mathcal{C}} \overline{\mathrm{T}}(I_C h) = \sum_{C \in \mathcal{C}} I_C \overline{\mathrm{T}}(I_C h)$ *for all* $h \in \mathcal{L}(\mathcal{X})$.

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1. $\overline{\mathrm{T}}(hI_B)(x) = 0$ *for all* $h \in \mathcal{L}(\mathcal{X})$*, all* $x \in C$ *and all* $B \subseteq C^c$*;*
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Consequently, if more communication classes exist, then PF-convergence is not possible.

Example

Assume there are n communication classes and take the gamble $f = \sum_{k=1}^{n} kI_{C_k}$ then $\overline{T}f = \sum_{k=1}^{n} I_{C_k} \overline{T} kI_{C_k} = \sum_{k=1}^{n} I_{C_k} \overline{T} k = f$.

Regularity implies PF-convergence

Definition

- A maximal aperiodic communication class is called regular.
- If there is only one communication class, then $\mathcal{X}$ is irreducible.
- If $\mathcal{X}$ is irreducible and aperiodic, $\mathcal{X}$ itself is also called regular.

$$(\exists n \in \mathbb{N})(\forall k \geq n)(\forall x, y \in \mathcal{X})(x \xrightarrow{k} y).$$

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$$(\exists n \in \mathbb{N})(\forall k \geq n)(\forall x, y \in \mathcal{X})(x \xrightarrow{k} y).$$

- It can be shown that regularity of $\mathcal{X}$ is a sufficient condition for PF-convergence.
- However, regularity is too strong.

Example

Take the precise model with transition matrix $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

Top class regularity

Definition

An imprecise Markov chain is said to be top class regular if

$$\mathcal{R}_{\rightarrow} := \left\{x \in \mathcal{X} : (\exists n \in \mathbb{N})(\forall k \geq n)(\forall y \in \mathcal{X})\overline{\mathrm{T}}^k I_{\{x\}}(y) > 0\right\} \neq \emptyset.$$

$\mathcal{R}_{\rightarrow}$ is the set of maximal regular states. This means that $\mathcal{R}$ is a regular top class whenever the Markov chain is regular.

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Top-class regularity is a necessary condition for PF-convergence, but not sufficient.

Example

Assume $\mathcal{X} = \{x, y\}$ and $\overline{T}f = \begin{pmatrix} f(x) \\ \max\{f(x), f(y)\} \end{pmatrix}$, then the $\mathbb{1}$ is belonging to the credal set to $\overline{T}$. Therefore there is no PF-convergence.

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Necessary and sufficient conditions for PF-convergence

Definition

A stationary imprecise Markov chain is called regularly absorbing if it is top class regular (under $\rightarrow$), meaning that

$$\mathcal{R}_{\rightarrow} := \left\{x \in \mathcal{X} \colon (\exists n \in \mathbb{N})(\forall k \geq n)(\forall y \in \mathcal{X})\overline{\mathrm{T}}^k I_{\{x\}}(y) > 0\right\} \neq \emptyset,$$

and if moreover it is leaky, i.e. for all y in $\mathcal{X} \setminus \mathcal{R}_{\rightarrow}$ there is some $n \in \mathbb{N}$ such that $\underline{\mathrm{T}}^n I_{\mathcal{R}_{\rightarrow}}(y) > 0$.

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Remark that accessibility to the complete communication class $\mathcal{R}$ is required and not to a state of $\mathcal{R}$.

Example

Assume $\mathcal{X} = \{x, y, z\}$ and $\underline{T}f = \begin{pmatrix} \min\{f(x), f(y)\} \\ \min\{f(x), f(y)\} \\ \min\{f(x), f(y)\} \end{pmatrix}$ then

$$\underline{T}^n I_{\{x\}} = \underline{T}^n I_{\{y\}} = 0 \quad \text{and} \quad \underline{T}^n I_{\{x,y\}} = I_{\{x,y\}}.$$

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Theorem

An imprecise Markov chain is PF-converging if and only if it is regularly absorbing.

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What about convergence in general?

Conjecture

An imprecise Markov chain converges if and only if

1. *the* $\longrightarrow$*-maximal communication classes are regular and*
2. *every periodic communication class leaks to the union of its* $\longrightarrow$*-dominating classes.*

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