Credal Sets and Reliability Bounds for Series Systems

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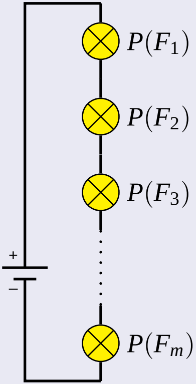
Series system of m components and system's reliability bounds



- F_i means that component i fails.
- $P(F_i)$ is the failure probability of component i.
- System fails if at least one component fails.

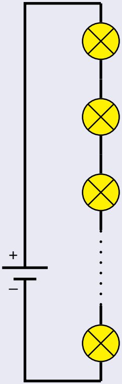
What is the probability p_f of failure for the system?

Reliability bounds for the system if nothing is known about dependencies between the components:

$$\max_{i=1,\dots,m} P(F_i) \le p_f \le \min\left(\sum_{i=1}^{m} P(F_i), 1\right)$$

(Fréchet bounds)

Series system of m components and system's reliability bounds



$$P(F_1) \in [\underline{P}(F_1), \overline{P}(F_1)]$$

$$P(F_2) \in [\underline{P}(F_2), \overline{P}(F_2)]$$

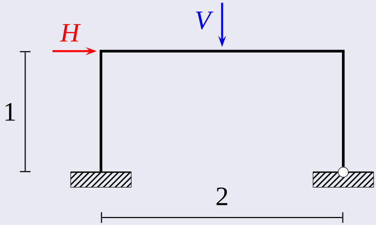
$$P(F_3) \in [\underline{P}(F_3), \overline{P}(F_3)]$$

$$P(F_m) \in [\underline{P}(F_m), \overline{P}(F_m)]$$

Extension:

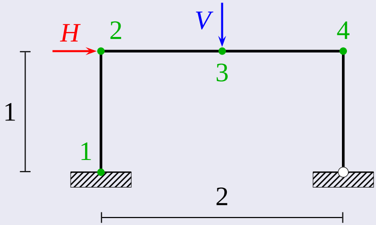
- Intervals given for the probabilities of failure of the components.
- Inserting the intervals into the formulas of the reliability bounds.
- See Lev Utkin's paper.

Rigid portal frame



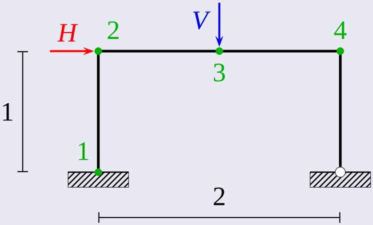
- Loads H and V.

Rigid portal frame



- Loads H and V.
- In 1, 2, 3, 4 plastic hinges may occur.
- Plastic moments M_1, M_2, M_3, M_4.

Rigid portal frame



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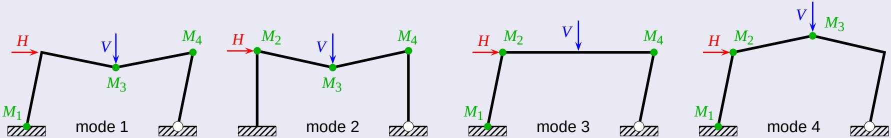
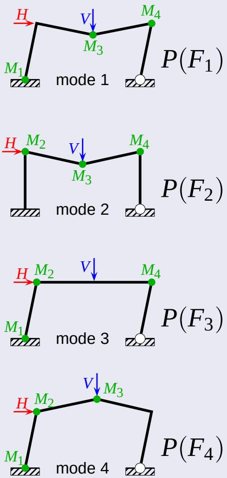
Four failure modes



Series system of m failure modes and system's reliability bounds



- F_i means that mode i occurs.
- $P(F_i)$ is the failure probability of mode i.
- System fails if at least one failure mode occurs.

What is the probability p_f of failure for the system?

Reliability bounds for the system if we want to decrease the computational effort:

$$\max_{i=1,\ldots,m} P(F_i) \leq p_f \leq \min\left(\sum_{i=1}^{m} P(F_i), 1\right)$$

Another Series System

Input variables

- Variables $X = (X_1, \ldots, X_n) = (M_1, M_2, M_3, M_4, H, V)$.

Modelling the uncertainty of the input variables X_i

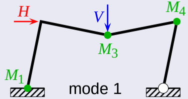
- Normal distributions. (What the engineers are doing)
- Parameterized probability distributions:

  Set $\mathcal{M}$ of all normal distributions with $\mu \in [\underline{\mu}, \overline{\mu}]$ and $\sigma \in [\underline{\sigma}, \overline{\sigma}]$.
- Set $\mathcal{M}$ of probability measures generated by p-boxes or random sets $\rightarrow$ credal set.
- We assume (strong or random set) independence.

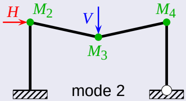
Limit state functions g_i

- For mode i: $g_i : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R} : \; x \mapsto g_i(x)$, $g_i(x) \leq 0 \rightarrow$ failure.

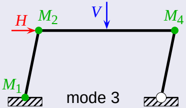
Limit state functions for the four failure modes



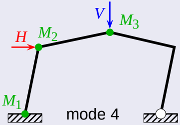
$$g_1(x) = M_1 + 2M_3 + 2M_4 - H - V$$



$$g_2(x) = M_2 + 2M_3 + M_4 - V$$



$$g_3(x) = M_1 + M_2 + M_4 - H$$



$$g_4(x) = M_1 + 2M_2 + 2M_3 - H + V$$

Another Series System

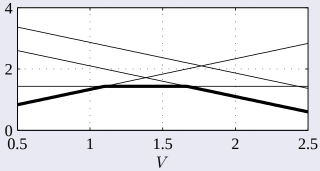
Limit state functions g_i and g_{syst}

- $F_i = g_i^{-1}((-\infty, 0])$.
- $g_{\text{syst}}(x) = \min_i g_i(x)$, probability of failure: $P(F_1 \cup \cdots \cup F_m)$.

  $P(F_1 \cup \cdots \cup F_m) \leq \min\left(\sum_{i=1}^{m} P(F_i), 1\right)$

Properties of g_i and g_{syst}

- $g(x) = (g_1(x), \ldots, g_m(x))^T = Ax$.
- g_i linear and monotonic (increasing or decreasing).
- g_{syst} non-linear, in general not monotonic.



Modelling the uncertainty of the variables

Each random variable X_i, $X = (M_1, M_2, M_3, M_4, H, V)$, is normally distributed, with parameters $(\mu_{X_i}, \sigma_{X_i})$:

$\mu_X = (1.0, 1.0, 1.0, 2.1, 2.0, 1.0)^T, \ \sigma_X = (0.15, 0.15, 0.15, 0.15, 0.17, 0.80)^T$.

Computation of $P(F_1)$, $P(F_2)$, $P(F_3)$, $P(F_4)$

The linear components $g_i(X)$ of $g(X)$ are again normally distributed with parameters $\mu_{g(X)} = \mathbf{A}\mu_X$ and $\sigma^2_{g(X)} = \mathbf{B}\sigma^2_X$ where

$$\mathbf{B}_{ij} = \mathbf{A}^2_{ij}, \quad \sigma^2_X = (\sigma^2_{X_1}, \ldots, \sigma^2_{X_6})^T, \quad \sigma^2_{g(X)} = (\sigma^2_{g_1(X)}, \ldots, \sigma^2_{g_4(X)})^T.$$

The first failure mode's failure probability, $P(F_1)$, is obtained as

$$P(F_1) = P(\{g_1(X) \leq 0\}) = F(0; \mu_{g_1(X)}, \sigma^2_{g_1(X)}) = F(0; \mathbf{A}_{1,*}\, \mu_X, \mathbf{B}_{1,*}\, \sigma^2_X)$$

where F is the value of the normal distribution function with parameters $\mu_{g_1(X)}$ and $\sigma^2_{g_1(X)}$, and evaluated at 0.

Numerical Example, Normal Distribution

Results

- $$P(F_1) = 3.4096 \cdot 10^{-6}\ , \quad P(F_2) = 1.6020 \cdot 10^{-6}$$
  $$P(F_3) = 6.7281 \cdot 10^{-12}, \quad P(F_4) = 9.1368 \cdot 10^{-6}\ .$$

- System reliability bounds:

$$p_f^- = \max_{i=1,\dots,4} P(F_i) = 9.1368 \cdot 10^{-6}$$

$$p_f^+ = \min \left(\sum_{i=1}^{4} P(F_i),\ 1 \right) = 1.4148 \cdot 10^{-5}.$$

- Using the limit state function g_{syst} and Monte-Carlo simulation the probability of failure of the system p_f is

$$p_f = P(\{g_{\text{syst}}(X) \leq 0\}) = 1.3138 \cdot 10^{-5}.$$

Questions

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- Strong or random set independence?

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- Is the computation of these intervals always cheaper than the computation of the probability of failure of the system using g_{syst}?
- Are there dependencies between the modes?
  Yes, because of shared variables.
- Strong or random set independence?
- Do the system reliability bounds help us if there is nothing known about how the variables interact?

Notations for the intervals of probabilities

$I_{F_i} = [\underline{P}(F_i), \overline{P}(F_i)]$ interval for the i-th mode's probability of failure,

$$\underline{P}(F_i) = \inf\{P(F_i) : P \in \mathcal{M}\},$$
$$\overline{P}(F_i) = \sup\{P(F_i) : P \in \mathcal{M}\},$$

$I_f = [\underline{p}_f, \overline{p}_f]$ interval for the system's probability of failure,

$I_{f,\mathrm{ex}}^- = [\underline{p}_{f,\mathrm{ex}}^-, \overline{p}_{f,\mathrm{ex}}^-]$ interval for the lower bound, exact computation,

$I_{f,\mathrm{ex}}^+ = [\underline{p}_{f,\mathrm{ex}}^+, \overline{p}_{f,\mathrm{ex}}^+]$ interval for the upper bound, exact computation,

$I_f^- = [\underline{p}_f^-, \overline{p}_f^-]$ interval for the lower bound, interval arithmetics,

$I_f^+ = [\underline{p}_f^+, \overline{p}_f^+]$ interval for the upper bound, interval arithmetics.

Computation of the bounds / problems

- If the intervals $I_{F_i} = [\underline{P}(F_i), \overline{P}(F_i)]$ are inserted into the formulas for the lower and upper system reliability bounds, upper bounds are overestimated and lower bounds are underestimated.
- Since the modes of failure share input variables X_i, there are interactions between the intervals I_{F_i}, $i = 1, \ldots, m$.
- By treating each interval separately, a repeated variable affecting two intervals is treated as if it were two different variables.
- The set of the probabilities of failure

$$S = \{(P(F_1), \ldots, P(F_m)) : \; P \in \mathcal{M}\}$$

is a subset of the Cartesian product of the failure probability intervals

$$S_{\Box} = I_{F_1} \times I_{F_2} \times \cdots \times I_{F_m}.$$

Exact bounds

$$\underline{p}^-_{f,\mathrm{ex}} = \min\left\{\max_{i=1,\ldots,m} P(F_i) : \ (P(F_1),\ldots,P(F_m)) \in S\right\},$$

$$\overline{p}^-_{f,\mathrm{ex}} = \max\left\{\max_{i=1,\ldots,m} P(F_i) : \ (P(F_1),\ldots,P(F_m)) \in S\right\},$$

$$\underline{p}^+_{f,\mathrm{ex}} = \min\left\{\min(\sum_{i=1}^{m} P(F_i), 1) : \ (P(F_1),\ldots,P(F_m)) \in S\right\},$$

$$\overline{p}^+_{f,\mathrm{ex}} = \max\left\{\min(\sum_{i=1}^{m} P(F_i), 1) : \ (P(F_1),\ldots,P(F_m)) \in S\right\}.$$

- In general, we have to solve two min-max optimization problems on the modes' probabilities of failure.

Approximate bounds

- Replacing S by $S_\square$ leads to interval arithmetics and to the formulas

$$\underline{p}_f^- = \max_{i=1,\dots,m} \underline{P}(F_i), \qquad \overline{p}_f^- = \max_{i=1,\dots,m} \overline{P}(F_i),$$

$$\underline{p}_f^+ = \min\left(\sum_{i=1}^m \underline{P}(F_i), 1\right), \qquad \overline{p}_f^+ = \min\left(\sum_{i=1}^m \overline{P}(F_i), 1\right)$$

- Outer approximations: $I_{f,\mathrm{ex}}^- \subseteq I_f^- = [\underline{p}_f^-, \overline{p}_f^-]$, $I_{f,\mathrm{ex}}^+ \subseteq I_f^+ = [\underline{p}_f^+, \overline{p}_f^+]$.
- Only for the more or less useless upper bound of the lower bound we have $\underline{p}_f^- = \underline{p}_{f,\mathrm{ex}}^-$, because interactions do not play a role in the calculation of $\max(\max(\cdot))$.
- In the following, we will also use the notation p_f^- for the lower bound $\underline{p}_f^-$ of the interval $[\underline{p}_f^-, \overline{p}_f^-]$, and p_f^+ for $\overline{p}_f^+$.

Conditions leading to exact bounds

- "Exact bounds" does not refer to the probability of failure for the system $I_f = [\underline{p}_f, \overline{p}_f]$. It refers to the exact intervals $I^-_{f,\mathrm{ex}}$ and $I^+_{f,\mathrm{ex}}$.
- In order to calculate the intervals $I^-_{f,\mathrm{ex}}$ and $I^+_{f,\mathrm{ex}}$ it is not required that $S = S_\Box$. It is sufficient to have

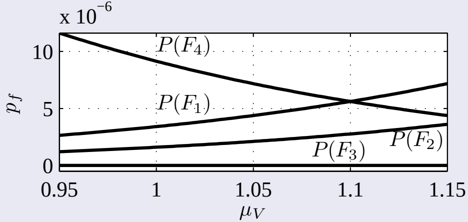
$$(\underline{P}(F_1), \ldots, \underline{P}(F_m)) \in S \quad \text{and} \quad (\overline{P}(F_1), \ldots, \overline{P}(F_m)) \in S,$$

  because these are the only values used in the above formulas.
- If credal sets are generated by random sets and if the limit state functions g_i are monotonic always in the same direction, then the above holds, because all $\underline{P}(F_i)$ and all $\overline{P}(F_i)$ can be obtained always at the same corners of the joint random sets.

Modelling the uncertainty of the variables

- Again
$$\mu_X = (1.0, 1.0, 1.0, 2.1, 2.0, *)^T,$$
$$\sigma_X = (0.15, 0.15, 0.15, 0.15, 0.17, 0.80)^T.$$
- The input and therefore the results are parameterized by the mean value μ_V of the vertical load V, $\mu_V \in [0.95,\ 1.15]$.
- $P(F_1)$, $P(F_2)$ and $P(F_3)$ are increasing functions in μ_V, but $P(F_4)$ is a decreasing function of μ_V.

Failure probabilities $P(F_1)$, $P(F_2)$, $P(F_3)$, $P(F_4)$ as functions of μ_V



Images of $[0.95,\ 1.15]$ under monotonic $P(F_1)$, $P(F_2)$, $P(F_3)$, $P(F_4)$

$$P(F_1) \in [\underline{P}(F_1), \overline{P}(F_1)] = [2.64662 \cdot 10^{-6}\ ,\ 7.17076 \cdot 10^{-6}\ ]$$
$$P(F_2) \in [\underline{P}(F_2), \overline{P}(F_2)] = [1.21401 \cdot 10^{-6}\ ,\ 3.61337 \cdot 10^{-6}\ ]$$
$$P(F_3) \in [\underline{P}(F_3), \overline{P}(F_3)] = [6.72815 \cdot 10^{-12}, 6.72815 \cdot 10^{-12}]$$
$$P(F_4) \in [\underline{P}(F_4), \overline{P}(F_4)] = [4.38048 \cdot 10^{-6}\ ,\ 1.16099 \cdot 10^{-5}\ ]$$

Approximate bounds p_f^- and p_f^+ using interval arithmetics

Inserting the above intervals into p_f^- and p_f^+:

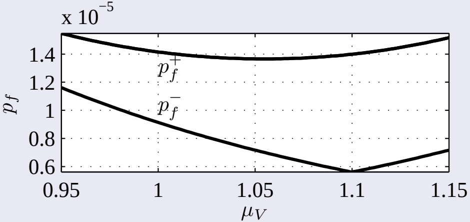
$$I_f^- = [4.38048 \cdot 10^{-6}\ ,\ 1.16099 \cdot 10^{-5}\ ] \supset I_{f,\text{ex}}^-$$
$$I_f^+ = [8.24112 \cdot 10^{-6}\ ,\ 2.23941 \cdot 10^{-5}\ ] \supset I_{f,\text{ex}}^+.$$

Exact bounds p_f^- and p_f^+, images of $[0.95,\ 1.15]$ under p_f^- and p_f^+

We calculate the exact bounds by computing the minimum and maximum of p_f^- and p_f^+ as functions of $\mu_V \in [0.95,\ 1.15]$:

$$I_{f,\mathrm{ex}}^- = [\underline{p}_{f,\mathrm{ex}}^-, \overline{p}_{f,\mathrm{ex}}^-] = [5.62629 \cdot 10^{-6}\ ,\ 1.16099 \cdot 10^{-5}\ ]$$
$$I_{f,\mathrm{ex}}^+ = [\underline{p}_{f,\mathrm{ex}}^+, \overline{p}_{f,\mathrm{ex}}^+] = [1.36547 \cdot 10^{-5}\ ,\ 1.54705 \cdot 10^{-5}\ ].$$

p_f^- and p_f^+ as functions of μ_V, non-linear, non-monotonic



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$$I_f^- = [4.38048 \cdot 10^{-6}\ ,\ 1.16099 \cdot 10^{-5}\ ]$$
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Modelling the uncertainty of the variables

$$\mu_{M_1} = \mu_{M_2} = \mu_{M_3} \in [0.75, 1.05],\ \mu_{M_4} \in [1.75, 2.2],$$

$H \in [1.9, 2.5]$, $V \in [0.75, 1.25]$ and

$$\sigma_X = (0.15, 0.15, 0.15, 0.15, 0.17, 0.80)^T.$$

Computation of $\underline{P}(F_i)$ and $\overline{P}(F_i)$

- $$\underline{P}(F_i) = F(0; \mathbf{A}_{i,*}\,\mu_X^{i+}, \mathbf{B}_{i,*}\,\sigma_X^2), \quad \overline{P}(F_i) = F(0; \mathbf{A}_{i,*}\,\mu_X^{i-}, \mathbf{B}_{i,*}\,\sigma_X^2)$$

  $$\mu_{X_j}^{i-} = \begin{cases} \mu_{X_j}^L & \mathbf{A}_{ij} > 0 \\ \mu_{X_j}^R & \mathbf{A}_{ij} < 0 \end{cases}, \qquad \mu_{X_j}^{i+} = \begin{cases} \mu_{X_j}^L & \mathbf{A}_{ij} < 0 \\ \mu_{X_j}^R & \mathbf{A}_{ij} > 0, \end{cases}$$

  if we assume (as in our example) that all mean values are positive.
- There are also rules for $\sigma_{X_i} \in [\sigma_{X_i}^L, \sigma_{X_i}^R]$.

Results

- $$P(F_1) \in [\underline{P}(F_1), \overline{P}(F_1)] = [7.64097 \cdot 10^{-8}\ , 1.60766 \cdot 10^{-2}\ ]$$
  $$P(F_2) \in [\underline{P}(F_2), \overline{P}(F_2)] = [8.69605 \cdot 10^{-8}\ , 8.92689 \cdot 10^{-4}\ ]$$
  $$P(F_3) \in [\underline{P}(F_3), \overline{P}(F_3)] = [5.38242 \cdot 10^{-15}, 7.85493 \cdot 10^{-3}\ ]$$
  $$P(F_4) \in [\underline{P}(F_4), \overline{P}(F_4)] = [4.15900 \cdot 10^{-7}\ , 1.60766 \cdot 10^{-2}\ ].$$
- Approximate system reliability bounds:
  $$p_f^- = 4.15900 \cdot 10^{-7}$$
  $$p_f^+ = 4.09007 \cdot 10^{-2}\ .$$

Truncated normal distributions

- The cumulative distribution function

$$F_{\text{trunc}}(x;\mu,\sigma^2) = \frac{F(x;\mu,\sigma^2) - F(x^L;\mu,\sigma^2)}{F(x^R;\mu,\sigma^2) - F(x^L;\mu,\sigma^2)}$$

  is the CDF which we get if a normal distribution with parameters μ, σ and CDF $F(x;\mu,\sigma^2)$ is truncated to the interval $[x^L, x^R]$.

- Start with lower and upper CDFs, $\underline{F}_i$ and $\overline{F}_i$, for each variable X_i:

$$\underline{F}_i(x) = F(x;\mu^R_{X_i},\sigma^2_{X_i}), \qquad \overline{F}_i(x) = F(x;\mu^L_{X_i},\sigma^2_{X_i}).$$

  (Means and variances from the previsious example)

- Replace $\underline{F}_i$ and $\overline{F}_i$ by the CDF of the corresponding truncated normal distributions.

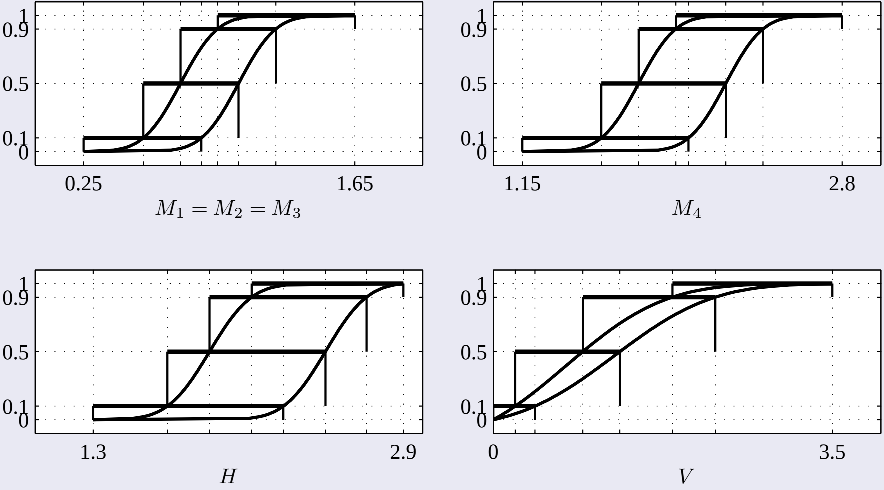
Truncation intervals for the variables

variable	interval for truncation of the lower CDF	interval for trunction of the upper CDF
M_1	$[0.25,\ 1.65]$	$[0.25,\ 1.65]$
M_2	$[0.25,\ 1.65]$	$[0.25,\ 1.65]$
M_3	$[0.25,\ 1.65]$	$[0.25,\ 1.65]$
M_4	$[1.15,\ 2.80]$	$[1.15,\ 2.80]$
H	$[1.30,\ 2.90]$	$[1.30,\ 2.90]$
V	$[0.00,\ 3.00]$	$[0.00,\ 3.50]$

- Approximation steps:

  Set of truncated normal distributions $\rightarrow$ p-box $\rightarrow$ random set.
  (Outer discretization method ODM, Fulvio Tonon)

Upper and lower CDFs of truncated normal distributions and random sets obtained by outer discretization for M_i, H, V



Computation of the images of the joint focal sets

- If random set independence is assumed, we have to compute the images
  $$B_i^j = [\underline{b}_i^j, \overline{b}_i^j] = g_i(A^j) \quad \text{and} \quad B_{\text{syst}}^j = [\underline{b}_{\text{syst}}^j, \overline{b}_{\text{syst}}^j] = g_{\text{syst}}(A^j)$$
  of all $4^6 = 4096$ joint random sets A^j.
- Mode's limit states g_i: Very easy because of monotonicity.
- Lower bounds, $\underline{b}_{\text{syst}}^j$, which are needed to calculate the upper probability:
  $$\underline{b}_{\text{syst}}^j = \min_{x \in A^j} g_{\text{syst}}(x) = \min_{x \in A^j} \min_i g_i(x) = \min_i \min_{x \in A^j} g_i(x) = \min_i \underline{b}_i^j.$$
- Upper bounds $\overline{b}_{\text{syst}}^j$ which are needed to calculate the lower probability:
  $$\overline{b}_{\text{syst}}^j = \max_{x \in A^j} g_{\text{syst}}(x) = \max_{x \in A^j} \min_i g_i(x) \leq \min_i \max_{x \in A^j} g_i(x) = \min_i \overline{b}_i^j.$$

Computation of the upper bounds $\overline{b}^{j}_{\mathrm{syst}}$

By solving the linear optimization problem

$$\text{maximize } y$$

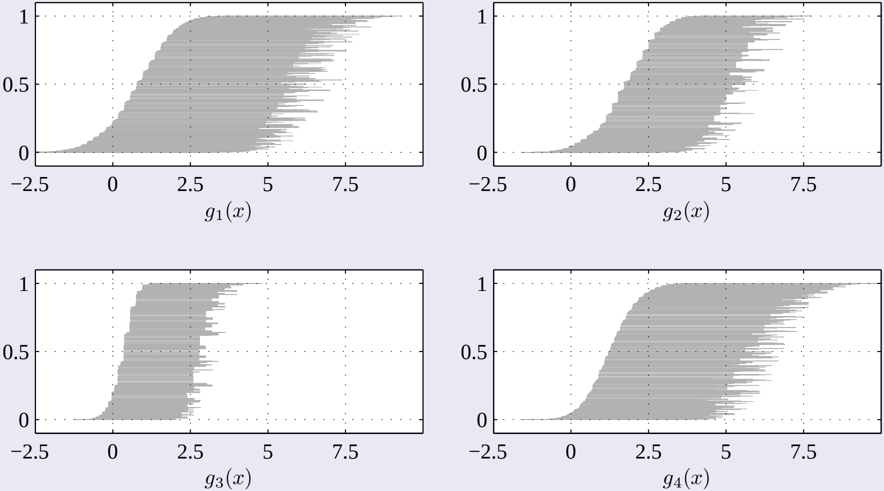
subject to

$$\begin{aligned} g_i(x) \geq y \quad & i = 1, \ldots, m \\ x_k \in I_k \quad & k = 1, \ldots, n \end{aligned}$$

where $I_1 \times \cdots \times I_m = A^j$ is the joint focal set generated by the Cartesian product of marginal focal sets (intervals) I_k.

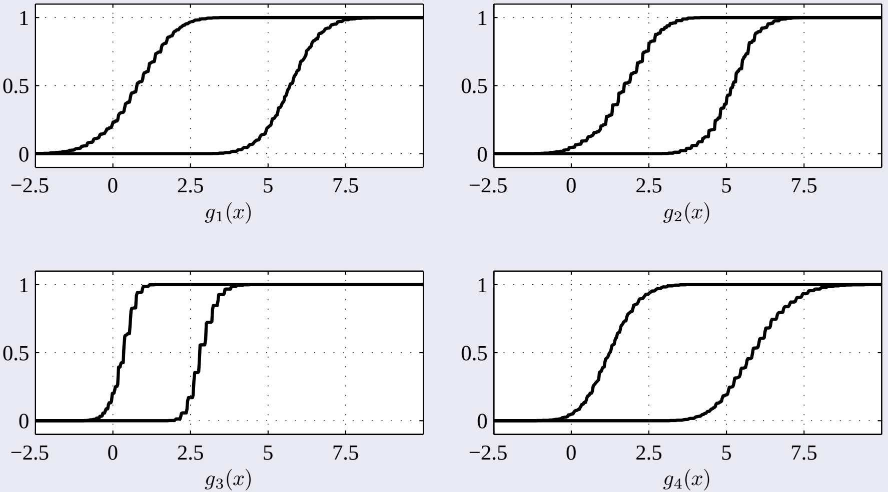
Numerical Example, Output (Coarse Discretization)

Images of the joint focal sets and p-boxes for the single modes



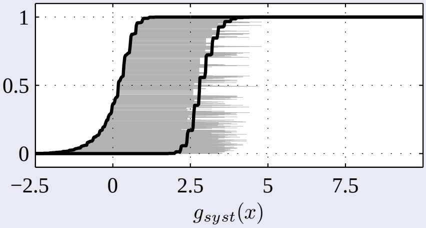
Numerical Example, Output (Coarse Discretization)

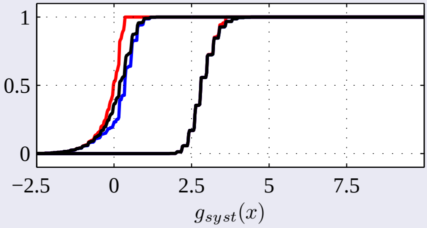
Images of the joint focal sets and p-boxes for the single modes



Numerical Example, Output (Coarse Discretization)

Images of the joint focal sets, p-box and bounds for the system





Numerical Example, Output (Coarse Discretization)

Results

- The probabilities of failure for the single failure modes:

$$P(F_1) \in [0,\ 2.319 \cdot 10^{-1}\ ]$$
$$P(F_2) \in [0,\ 4.410 \cdot 10^{-2}\ ]$$
$$P(F_3) \in [0,\ 2.021 \cdot 10^{-1}\ ]$$
$$P(F_4) \in [0,\ 4.938 \cdot 10^{-2}\ ].$$

- Approximate system reliability bounds:

$$p_f^- = 0, \quad p_f^+ = 5.27480 \cdot 10^{-1}\ .$$

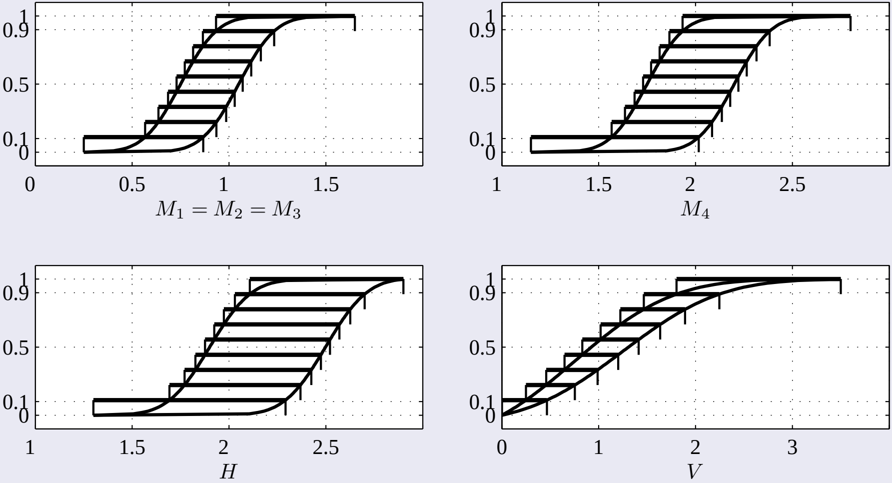
- The system's probability of failure obtained using g_{syst}:

$$p_f \in [0,\ 3.65221 \cdot 10^{-1}\ ].$$

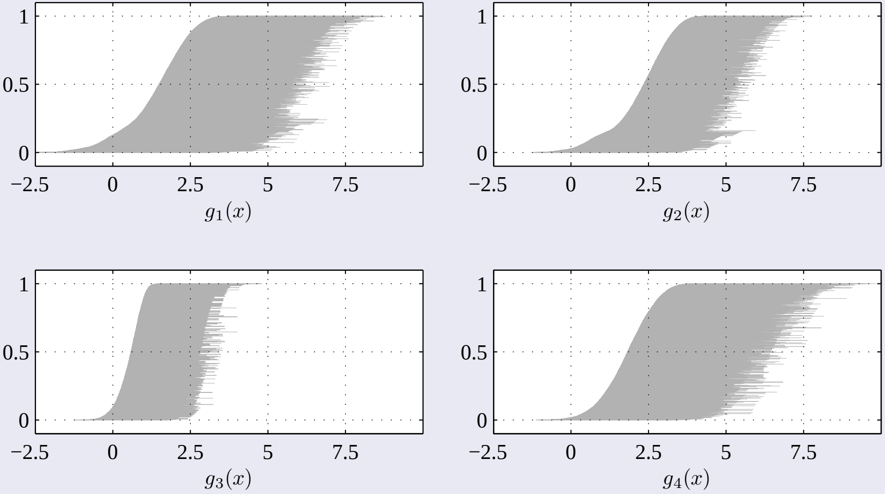
Monte-Carlo

- Using only four focal sets leads to a very rough approximation of the p-boxes.
- If we use a finer discretization, e.g., 10 focal sets, we would get a better approximation, but then we have to compute 10^6 images of joint focal sets. The idea is now not to consider all 10^6 joint focal sets, but only, say, $N = 10,000$ randomly chosen sets.
- Notice: Probability bounds are no longer automatically verified.
- Algorithm:
  1. For each variable x_k choose N focal sets according to the weights m_k.
  2. The j-th joint focal set is the Cartesian product of all j-th chosen marginal focal sets, $j = 1, \ldots, N$.
  3. The weights of these joint focal sets are $1/N$.

Upper and lower CDFs of truncated normal distributions and random sets obtained by outer discretization for M_i, H, V

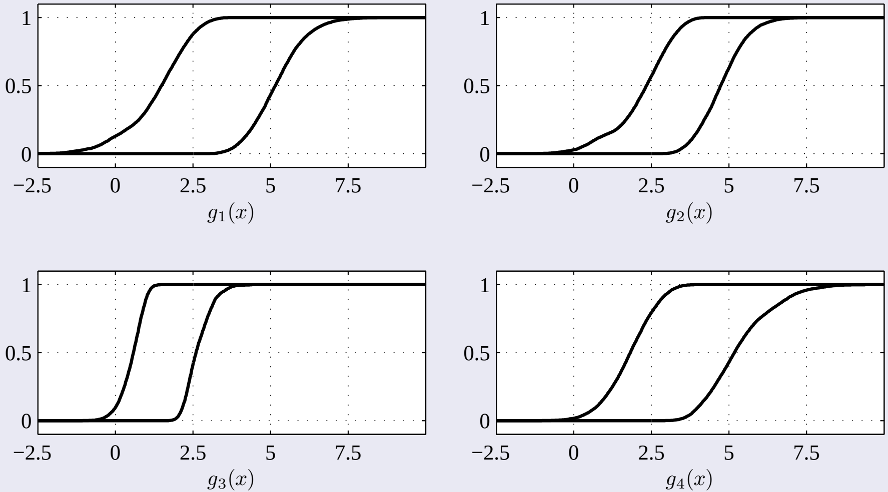


Images of the joint focal sets and p-boxes for the single modes

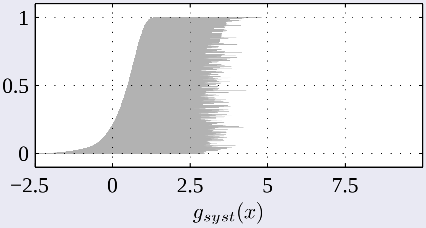


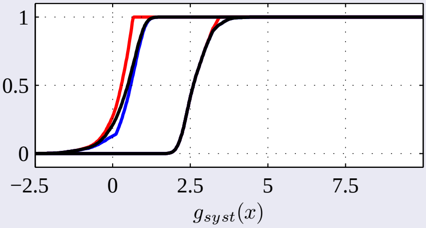
Numerical Example, Output (Fine Discretization, Monte-Carlo)

Images of the joint focal sets and p-boxes for the single modes



Images of the joint focal sets, p-box and bounds for the system





Results for Monte-Carlo Simulation, $N = 10,000$

Probabilities of failure for the single failure modes	
Discretization: 10 focal sets	$10,000$ focal sets
$P(F_1) \in [0,\ 1.251 \cdot 10^{-1}\ ]$ $P(F_2) \in [0,\ 2.670 \cdot 10^{-2}\ ]$ $P(F_3) \in [0,\ 9.680 \cdot 10^{-2}\ ]$ $P(F_4) \in [0,\ 1.710 \cdot 10^{-2}\ ]$	$P(F_1) \in [0,\ 1.45 \cdot 10^{-2}\ ]$ $P(F_2) \in [0,\ 3.00 \cdot 10^{-4}\ ]$ $P(F_3) \in [0,\ 6.30 \cdot 10^{-3}\ ]$ $P(F_4) \in [0,\ 1.00 \cdot 10^{-4}\ ]$

The system reliability bounds	
$p_f^- = 0$ $p_f^+ = 2.657 \cdot 10^{-1}$	$p_f^- = 0$ $p_f^+ = 2.12 \cdot 10^{-2}$

The system's probability of failure obtained using g_{syst}	
$p_f \in [0,\ 2.042 \cdot 10^{-1}\ ]$	$p_f \in [0,\ 2.04 \cdot 10^{-2}\ ]$

Recalling the main reason for using system reliability bounds

- Linear limit state functions $g_1, \ldots, g_m$ for the failure modes.
- Variables $X_1, \ldots, X_n$ normaly distributed.

$\rightarrow$ Easy computation of $P(F_1), \ldots, P(F_m)$.

- Non-linear and non-monotonic limit state function g_{syst}.

$\rightarrow$ High computional effort (compared to the failure modes).

Linear g_i, monotonicity always in the same direction

- Parameterized probabilities (normal distribution):
  - Single mode: Low effort.
  - System (non-linear): High effort.

$\rightarrow$ Use system reliability bounds (exact bounds).

- Random sets, p-boxes:
  - Single mode: Low effort.
  - System (monotonic): Low effort.
  - Random set independence $=$ strong independence.

$\rightarrow$ Do not use system reliability bounds.

Linear g_i, monotonicity not always in the same direction

- Parameterized probabilities (normal distribution):
  - Single mode: Low effort.
  - System (non-linear): High effort.

  $\rightarrow$ Use system reliability bounds (approximate bounds only).
- Random sets, p-boxes:
  - Single mode: Low effort.
  - System (non-monotonic)
    - Upper probability: Low effort.
    - Lower probability: More expensive (linear program).
  - Random set independence $\neq$ strong independence.
  - Approximate bounds only.